

Queen Mary, University of London

B.Sc. EXAMINATION

MAS204 Calculus III

28 May 2002 10 AM - 1 PM

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination.

You are reminded of the following, which you may use without proof.
In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = r \sin \theta$.

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = 1$.

SECTION A

Each question carries 8 marks. You should attempt ALL questions.

- A1.** (a) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ along the curve C on which $(x, y, z) = (t, t, 2t^2)$ from $t = 0$ to $t = 1$.
(b) Calculate ∇f for $f = xz + xy^2$ and evaluate it at $(1, 3, -2)$.

- A2.** Calculate $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for

$$\mathbf{F} = r \sin \theta \mathbf{e}_r + r \cos \theta \mathbf{e}_\theta + z \cos \theta \mathbf{e}_z,$$

in cylindrical polar coordinates.

- A3.** Using index notation, or otherwise,

- (a) evaluate $\nabla \cdot (\mathbf{a} \times \mathbf{r})$ where $\mathbf{a}$ is a constant vector and $\mathbf{r}$ is the usual position vector (with components (x, y, z) in Cartesian coordinates); and
(b) prove that $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$.

[You may assume the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]

- A4.** State Stokes's Theorem.

Using this theorem, or otherwise, evaluate $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$ over the surface S of the hemisphere defined by $x^2 + y^2 + z^2 = 4$, $z > 0$, where the z -component of $d\mathbf{S}$ is non-negative on S .

- A5.** For each of the following vector fields either find the general solution V of $\mathbf{F} = \nabla V$ or give a reason why no such V exists.

- (a) $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} + (y + 1)\mathbf{k}$.
(b) $\mathbf{F} = \mathbf{e}_\phi$ in spherical polar coordinates.

- A6.** Show that at a general point the following are solutions of Laplace's equation $\nabla^2 V = 0$.

- (a) $V = \frac{\sin n\theta}{r^n}$, for an integer n , in cylindrical polar coordinates.
(b) $V = \frac{\sin \theta \sin \phi}{r^2}$, in spherical polar coordinates.

- A7.** Find the Fourier series of the function $f = x^2$ on the range $(-\pi, \pi)$.

SECTION B

Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

B1. State the Divergence Theorem carefully.

In spherical polar coordinates the vector field $\mathbf{F}$ is given by

$$\mathbf{F} = r \sin \theta (\mathbf{e}_r + \mathbf{e}_\theta + \mathbf{e}_\phi).$$

Verify the Divergence Theorem for this vector field over a sphere of radius a centered at the origin.

Using the Divergence Theorem or otherwise, evaluate $\int \mathbf{F} \cdot d\mathbf{S}$ over the surface of the hemisphere $x^2 + y^2 + z^2 \leq a^2, z \geq 0$.

B2. The function $f(x) = x$ in the range $0 < x < \pi$. Find the Fourier sine series and Fourier cosine series for $f(x)$. Sketch the functions to which these two series converge on the range $[-2\pi, 2\pi]$.

By adding the two series obtained, or otherwise, show that the Fourier series for the function $g(x)$ defined by $g = 0$ on $-\pi < x \leq 0$ and $g = f$ on $0 < x < \pi$ is

$$g(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}.$$

Hence show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}, \quad \text{and} \quad \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)} = \frac{\pi}{4}.$$

B3. Consider Laplace's equation $\nabla^2 V = 0$ in the case where V is a function of Cartesian coordinates x and y only. Find all possible separable solutions of the form $V(x, y) = X(x)Y(y)$. Describe the form of the general solution.

Find the function $V(x, y)$ satisfying $\nabla^2 V = 0$ in the region $0 < x < a, 0 < y < b$ and subject to the boundary conditions $V = \cos(\pi x/a)$ on $y = 0$ and $V = 0$ on the other three sides of the region. (You may assume that the Fourier sine series for $\cos(\pi x/a)$ is

$$\cos(\pi x/a) = \sum_{k=1}^{\infty} \frac{8k \sin(2k\pi x/a)}{(4k^2 - 1)\pi}$$

on $0 < x < a$.)

Explain briefly why and how this result can be used to obtain the solution to the following problem, and write down the resulting $V(x, y)$.

Find $V(x, y)$ satisfying $\nabla^2 V = 0$ in the region $0 < x < a$, $0 < y < b$ and subject to the boundary conditions $V = K \cos(\pi x/a)$ on $y = 0$, $V = M \cos(\pi y/b)$ on $x = 0$, and $V = 0$ on the other two sides of the region, where K and M are constants.

B4. Show that if $y(x)$ maximizes or minimizes the integral

$$I[y] = \int_{x_0}^{x_1} F(x, y, y') \, dx ,$$

with $y(x_0)$ and $y(x_1)$ fixed and F given, then y satisfies the Euler-Lagrange equation

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0 .$$

Show that

- (a) if $F = F(y, y')$ (i.e. F has no explicit x dependence) then the Euler-Lagrange equation has a first integral

$$F - y' \frac{\partial F}{\partial y'} = \text{constant} ;$$

- (b) if $F = F(x, y')$ (i.e. F has no explicit y dependence) then the Euler-Lagrange equation has a first integral

$$\frac{\partial F}{\partial y'} = 0 .$$

A light ray in a medium of variable composition has speed $v(y) = \cos y$ in the range $0 \leq y \leq \pi/4$. Show that (assuming light travels on the path of shortest time) the ray between the points $(0, \pi/6)$ and $(1, 0)$ in the (x, y) plane satisfies

$$2 \sinh 1 \sin y = \sinh(1 - x).$$

[Hint: for the final integration you may find a substitution of the form $\sin y = B \sinh w$, for some constant B , useful]